ISHIKAWA ITERATIVE SCHEME FOR LIPSCHITZIAN PSEUDOCONTRACTIONS

(Short Communication)

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Abstract: *In this note, we establish the strong convergence for the Ishikawa iterative scheme associated with Lipschitzian pseudocontractive mappings in Hilbert spaces. The remark at the end is important.*

Key Words Ishikawa iterative scheme, Lipschitzian mappings, Pseudocontractive mappings, Hilbert spaces

Main Results

In 1974, Ishikawa [1] introduced an iteration scheme which, in some sense, is more general than that of Mann [2] and proved the following results.

Theorem 1 *If K is a compact convex subset of a Hilbert space H, $T : K \mapsto K$ is a Lipschitzian pseudocontractive mapping and x_1 is any point in K, then the sequence $\{x_n\}_{n\geq 1}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 1$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n,$$

and $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences of positive numbers satisfying the conditions

$$(i) 0 \leq \alpha_n \leq \beta_n < 1; (ii) \lim_{n\to\infty} \beta_n = 0;$$

$$(iii) \sum_{n\geq 1} \alpha_n \beta_n = \infty.$$

However we restate the above theorem as follows.

Theorem 2 *If K is a compact convex subset of a Hilbert space H, $T : K \mapsto K$ is a Lipschitzian pseudocontractive mapping satisfying*

$$\|x - Ty\| \leq \|Tx - Ty\| \text{ for all } x, y \in K, \tag{C}$$

and x_1 is any point in K, then the sequence $\{x_n\}_{n\geq 1}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 1$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n,$$

and $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences of positive numbers satisfying the conditions

$$(i) \sum \alpha_n \beta_n = \infty, (ii) \lim_{n\to\infty} \beta_n = 0.$$

As a particular case, we may choose for instance

$$\alpha_n = \frac{n}{n+1}, \beta_n = \frac{1}{n}.$$

Proof. Let p denote any point of $F(T)$. Following the lines of Ishikawa [1], by using condition (C) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - \alpha_n \beta_n \\ &\quad (1 - 2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + \alpha_n \beta_n \|Tx_n - Ty_n\|^2 - \alpha_n(\beta_n - \alpha_n)\|x_n - Ty_n\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_n \beta_n (1 - 2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + \alpha_n \beta_n \|Tx_n - Ty_n\|^2 + \alpha_n^2 \|x_n - Ty_n\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_n \beta_n (1 - 2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + \alpha_n(\alpha_n + \beta_n)\|Tx_n - Ty_n\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_n \beta_n (1 - 2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + 2\alpha_n \|Tx_n - Ty_n\|^2 \end{aligned} \tag{1}$$

Also since T is Lipschitzian,

$$\|Tx_n - Ty_n\| \leq L\beta_n \|x_n - Tx_n\|,$$

and (1) implies

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - \alpha_n \beta_n \\ &\quad (1 - 2(1+L^2)\beta_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{2}$$

Now by (ii), $\lim \beta_n = 0$ implies that there exists $n_0 \in \mathrm{N}$ such that for all $n \geq n_0$,

$$\beta_n \leq \frac{1}{4(1+L^2)},$$

and also (2) yields

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \frac{1}{2}\alpha_n \beta_n \|x_n - Tx_n\|^2,$$

implies

$$\frac{1}{2}\alpha_n \beta_n \|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

so that

$$\frac{1}{2}\sum_{j=m}^{n} \alpha_j \beta_j \|x_j - Tx_j\|^2 \leq \|x_m - p\|^2 - \|x_{n+1} - p\|^2.$$

The rest of the argument follows exactly as in the proof of the main Theorem of [1].

Remark 1 *This kind of reconstruction is new under the setting of Hilbert spaces.*

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